Horizon Thermality from Pure Geometry: A Coordinate-Free Derivation

Ira Wolfson
Department of Electrical and Electronic Engineering,
Braude Academic College of Engineering, Karmiel, Israel[*]

We derive the Hawking-Unruh thermal spectrum $|\beta_\omega|^2 = (e^{2\pi\omega/\kappa} - 1)^{-1}$ from three geometric ingredients: a smooth Lorentzian manifold carrying a null hypersurface $\mathcal{H}$, a vector field ξ^μ with inaffinity κ on $\mathcal{H}$, and wave-like dynamics with normalizable modes. No gravitational field equations, no specific metric, and no global interior geometry is invoked. The thermal factor arises as the unique admissible analytic extension of flow-adapted modes past the coordinate sign-change at $\mathcal{H}$, selected by the positive-frequency condition. The Bogoliubov map between flow-adapted and null-adapted bases is a non-compact element of $\mathrm{Sp}(2,\mathbb{R})$ — a symplectic squeeze — whose classical analogue is the Coriolis effect. The Unruh effect is the smooth case: two time foliations, no singular surface. Hawking radiation is the singular case: the flow-adapted foliation becomes singular at $\mathcal{H}$ because ξ^μ goes null there. The spectrum is theory-independent: any theory of gravity admitting a null horizon with inaffinity κ and standard local hyperbolic wave propagation predicts $T = \kappa/2\pi$. The inverse-temperature four-vector $\beta^\mu = 2\pi\xi^\mu/\kappa$ emerges as the covariant output.

Introduction.— Hawking's original derivation [1] works in Schwarzschild coordinates, invokes the Einstein equations to fix the geometry, and computes Bogoliubov coefficients by mode matching across the horizon. The result appears to depend on the details of general relativity. It does not.

We show that the Planck spectrum $|\beta_\omega|^2 = (e^{2\pi\omega/\kappa} - 1)^{-1}$ follows from three ingredients alone: a null hypersurface $\mathcal{H}$, a vector field ξ^μ with inaffinity κ, and wave-like dynamics. While coordinate-independent and theory-independent approaches to horizon thermality exist [4, 5], the derivation presented here isolates a more primitive geometric ingredient — the inaffinity eigenvalue equation — and obtains the Planck factor through a proof architecture not previously used. The derivation makes four contributions beyond the standard treatment. First, the logarithmic relation $\tau = \kappa^{-1}\ln\xi$ — the source of the branch structure — is derived directly from the inaffinity eigenvalue equation $\xi^\nu\nabla_\nu\xi^\mu = \kappa\xi^\mu$, with no metric, no coordinate expansion, and no near-horizon approximation. Second, the Bogoliubov coefficients emerge as symplectic projections onto a complete null-adapted basis — the exact analogue of Fourier coefficient extraction — so the thermal factor is determined by the analytic structure of the admissible mode before any projection is performed. Third, the result is strictly theory-independent: any theory of gravity with standard local hyperbolic wave propagation predicts $T = \kappa/2\pi$; GR enters only to fix $\kappa(M)$. Fourth, the inverse-temperature four-vector $\beta^\mu = 2\pi\xi^\mu/\kappa$ emerges as the covariant output, not a postulate.

Geometric setup.— Let $\mathcal{H} \subset M$ be a null hypersurface. Its null normal ℓ^μ satisfies $\ell^\mu\ell_\mu = 0$ — simultaneously normal and tangent to $\mathcal{H}$. The flow along ℓ^μ stays on $\mathcal{H}$; it never crosses it.

Let ξ^μ be a smooth vector field on M satisfying: (i) ξ^μ maps $\mathcal{H}$ to itself; (ii) $\xi^\mu\xi_\mu < 0$ in M_+; (iii) $\xi^\mu\xi_\mu = 0$ on $\mathcal{H}$; (iv) the flow of ξ^μ preserves the symplectic form σ on the solution space $\mathcal{S}$.[10]

Foliate $\mathcal{H}$ by the flow of ξ^μ. The leaves are 2-surfaces of constant flow-parameter τ. As $\tau \to \pm\infty$ they degenerate to the fixed-point set:

$$\mathcal{B} = \{p \in \mathcal{H} : \xi^\mu(p) = 0\}. \tag{1}$$

By Stone's theorem [3], the time-translation representation of $(\mathbb{R}, +)$ is of the form $e^{-i\omega\tau}$ in any coordinate system. The inaffinity equation $\xi^\nu\nabla_\nu\xi^\mu = \kappa\xi^\mu$ on $\mathcal{H}$ is an eigenvalue equation: ξ^μ is an eigenvector of $\xi^\nu\nabla_\nu$ with eigenvalue κ [4]. Along the transverse flow this gives $d\xi^\mu/d\tau = \kappa\xi^\mu$, so $|\xi^\mu(\tau)| \sim e^{\kappa\tau}$. Defining $\xi \equiv |\xi^\mu|$:

Lemma (Logarithmic flow). For any vector field ξ^μ satisfying the inaffinity equation on a null hypersurface, the flow parameter satisfies

$$\tau = \frac{1}{\kappa}\ln\xi. \tag{2}$$

No metric, no coordinate choice — just the eigenvalue κ. The Stone representation follows directly: $e^{-i\omega\tau} = \xi^{-i\omega/\kappa}$. This logarithmic relation forces branch-structured flow modes. The Planck spectrum is its quantum corollary.

Solution space and two bases.— Physical information propagates via wave-like dynamics. We model the field by any object ϕ — scalar, vector, spinor, tensor — satisfying a linear wave equation on M. The covariant derivative ∇_μ acts on ϕ, responding to its tensor structure. The solution space $\mathcal{S}$ is a real vector space carrying the symplectic form [4]:

$$\sigma(f, g) = \int_\Sigma (f\nabla_\mu g - g\nabla_\mu f)\, n^\mu\, d\Sigma, \tag{3}$$

independent of the Cauchy surface Σ. Linearity is sufficient but not necessary — the actual requirement is that $(\mathcal{S}, \sigma)$ admits two complete non-degenerate bases related by an invertible map. σ is defined to play the role of the

Poisson bracket: it identifies canonical pairs and detects the positive/negative frequency split that defines a_k and $a_k^\dagger$ upon quantization.

We choose two physically meaningful bases. *Basis 1* (flow-adapted): modes $u_\omega \sim e^{-i\omega\tau} = \xi^{-i\omega/\kappa}$ — singular at $\xi = 0$, natural for the exterior observer. *Basis 2* (null-adapted): modes $v_\Omega \sim e^{-i\Omega\lambda}$, where λ is the affine parameter of the null generators of $\mathcal{H}$ — regular everywhere, no reference to any observer. The inaffinity equation requires affine reparametrization satisfying $d^2\lambda/d\tau^2 = \kappa\, d\lambda/d\tau$, giving $\lambda \sim e^{\kappa\tau}$.

Let $S : \mathcal{S} \to \mathcal{S}$ express each v_Ω in terms of $\{u_\omega\}$. Since f and g are geometric objects independent of basis, $\sigma(Sf, Sg) = \sigma(f, g)$ automatically — S is a symplectic automorphism of $(\mathcal{S}, \sigma)$.

The thermal spectrum.— The flow-adapted modes do not span all of $(\mathcal{S}, \sigma)$ — they are cut off at $\xi = 0$. At $\xi = 0$ the coordinate simply changes sign; $\xi \to -\xi$ gives two possible extensions:

$$u_\omega^\pm = |\xi|^{-i\omega/\kappa} \cdot e^{\pm\pi\omega/\kappa}. \quad (4)$$

The $(+)$ branch is excluded by the positive-frequency admissibility condition: physical modes must have finite norm in the complex structure induced on $(\mathcal{S}, \sigma)$ by the flow of ξ^μ in M_+.[11] We keep $e^{-\pi\omega/\kappa}$.

The Bogoliubov coefficients are the symplectic projections of u_ω onto the null-adapted basis:

$$\alpha_{\omega\Omega} = \sigma(u_\omega, v_\Omega), \qquad \beta_{\omega\Omega} = -\sigma(u_\omega, v_\Omega^*). \quad (5)$$

This is exactly the structure of Fourier coefficient extraction: given a complete basis and a function, the coefficients are uniquely determined by projection. In particular, since the inadmissible branch u_ω^+ is excluded from the physical solution space, it enters with coefficient zero: $u_\omega = u_\omega^- + 0 \cdot u_\omega^+$. The admissible mode u_ω^- carries the relative amplitude $e^{-\pi\omega/\kappa}$ between its $\xi < 0$ and $\xi > 0$ pieces. The oscillatory factor $|\xi|^{-i\omega/\kappa}$ has unit modulus and cancels in the ratio, so the symplectic projection directly gives:

$$\frac{|\beta_\omega|}{|\alpha_\omega|} = e^{-\pi\omega/\kappa}. \quad (6)$$

The symplectic constraint $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$ then gives:

$$\boxed{|\beta_\omega|^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1}.} \quad (7)$$

Corollary (Planck spectrum). For any smooth Lorentzian manifold containing a non-extremal null hypersurface $\mathcal{H}$ with generator ξ^μ of inaffinity κ, the admissible analytic continuation of flow-adapted modes $u_\omega \sim \xi^{-i\omega/\kappa}$ implies Bogoliubov coefficients satisfying $|\beta_\omega|^2 = (e^{2\pi\omega/\kappa} - 1)^{-1}$: a Planck distribution at $T = \kappa/2\pi$, from the logarithmic flow lemma, positive-frequency admissibility, and symplectic projection. No field equations. No global interior geometry.

Interpretation.— The spectrum reflects the mismatch between two valid mode decompositions of the same field. The classical analogue is the Coriolis effect: fictitious forces from insisting on a rotating-frame basis. The *Unruh effect* [2] is the smooth quantum version — two time foliations, no singular surface, all temperature concepts finite. *Hawking radiation* is the singular case: ξ^μ goes null at $\mathcal{H}$, the basis-shifting map becomes physically incomplete, and for a collapsing star the time-dependent mismatch appears as real energy flux at infinity. The two transformations are elements of $\mathrm{Sp}(2, \mathbb{R})$: Unruh/Coriolis is compact (rotation, removable mismatch); Hawking is non-compact (squeeze, permanent mismatch). The covariant output of the derivation is the inverse-temperature four-vector [6, 7] $\beta^\mu = 2\pi\xi^\mu/\kappa$, whose null degeneration at $\mathcal{H}$ and connection to Susskind's temperature [8] are developed in a companion paper [9].

Minkowski projection.— For Schwarzschild, $\xi^\mu = (\partial_t)^\mu$ and $\kappa = c^4/4GM$:

$$T_0 = \frac{\hbar c^3}{8\pi G M k_B} \approx \frac{6 \times 10^{-8}}{M/M_\odot}\ \mathrm{K}. \quad (8)$$

Hawking's result [1] is the corollary of specifying GR and reading off $\kappa(M)$. The present derivation is strictly more general.

Conclusions.— The Hawking-Unruh thermal spectrum is a forced consequence of the geometry of any non-extremal null surface: the logarithmic singularity of the exterior flow time, forced by the inaffinity eigenvalue equation, produces branch-structured modes whose admissible continuation uniquely fixes the Planck factor by symplectic projection. The result holds for any field and any theory of gravity with standard local hyperbolic wave propagation. The primitive ingredient is not Einstein gravity — it is the inaffinity κ of a null surface generator.

* wolfsoni@braude.ac.il

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[10] Our derivation requires only that κ be defined locally. The combination of conditions (i)–(iii) is consistent with κ being constant across all of $\mathcal{H}$ — the zeroth law of black hole thermodynamics [4] — though establishing this fully

requires additional smoothness assumptions beyond the scope of this letter.
[11] This is the field-theoretic analogue of the outgoing radiation condition in scattering theory. It selects the physically propagating branch.