Geometric Origin of the Bekenstein–Hawking Entropy Prefactor from Null-Boundary Phase Space

Ira Wolfson[1]

[1] *Department of Electrical and Electronic Engineering, Braude Academic College of Engineering, Karmiel 2161002, Israel*

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The Bekenstein–Hawking entropy $S = k_B A/(4\ell_P^2)$ is commonly obtained from semiclassical thermodynamics or reproduced by microscopic state counts in candidate quantum gravity theories. We show that the numerical prefactor 1/4 can be traced to the covariant phase-space structure of general relativity in the presence of a null boundary. Interpreting entropy as the logarithm of accessible Liouville phase-space volume, we analyze the null-boundary symplectic structure relevant to observers restricted to the exterior of a causal horizon. Two geometric reductions act on the null data: (i) the null characteristic constraint structure (Raychaudhuri/Damour) restricts the kinematic null data to a constraint surface, and (ii) consistency of an exterior observable algebra induces a symplectic quotient by directions invisible to all exterior-supported variations. These reductions fix the effective symplectic rank density of the exterior phase space, leading directly to the Bekenstein–Hawking normalization as a geometric property of gravitational phase space rather than a model-dependent microscopic input.

I. INTRODUCTION

Black hole entropy obeys [1–3]

$$S = \frac{k_B A}{4\ell_P^2}. \tag{1}$$

Semiclassical derivations based on Hawking radiation and Euclidean methods reproduce this formula, and microscopic counts in proposed quantum gravity theories can match the same coefficient. However, those approaches do not isolate whether the factor 1/4 reflects a more general geometric feature of gravitational phase space or merely inherits the conventional normalization of the Einstein–Hilbert action.

An additional structural reason to look beyond Hamiltonian thermodynamic arguments arises at null boundaries themselves. Standard thermodynamic derivations rely on a Hamiltonian framework: spacetime is foliated by spacelike hypersurfaces, canonical data are evolved between time slices, and energy generates time translations. Relations such as $\delta E = T\,\delta S$ therefore depend on the existence of a well-defined notion of time evolution.

At a null surface this structure degenerates. The hypersurface is tangent to the light cones, the lapse vanishes, and evolution along the null generators is governed by characteristic constraints rather than by a Hamiltonian evolution between Cauchy slices. Consequently, entropy associated with a horizon cannot fundamentally originate from Hamiltonian flow at the boundary itself; rather, it must already be encoded in the covariant phase-space geometry of the null boundary. This motivates a direct symplectic analysis of null gravitational data independent of thermodynamic energy arguments.

Here we provide such a phase-space-based route.

II. ENTROPY AS LIOUVILLE MEASURE

For a Hamiltonian system,

$$S = k_B \ln \Omega_{\mathrm{acc}}, \tag{2}$$

where Ω_{acc} is the accessible Liouville phase-space volume [11, 12]. Throughout this work logarithms are natural unless stated otherwise.[1]

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[1] Macroscopic thermodynamic entropy is expressed in natural logarithms. While information entropy may be defined in other bases (bits, etc.), Stirling's approximation for large numbers of microscopic degrees of freedom yields natural logarithms, making thermodynamic entropy independent of the chosen microscopic information base.

III. NULL-BOUNDARY SYMPLECTIC STRUCTURE

We use the covariant phase-space formulation [4, 5]. Pulling the Lee–Wald symplectic potential to a null hypersurface $\mathcal{N}$ yields [6, 7]

$$\Omega = \frac{1}{16\pi G} \int_{\mathcal{N}} \Big(\delta\pi^{AB} \wedge \delta h_{AB} + \delta\eta \wedge \delta A + \cdots \Big), \tag{3}$$

A key geometric point is that the fundamental cell in this counting is an element of the two-dimensional horizon cross-section Σ, not a three-dimensional spatial volume element. A spacelike Cauchy slice in $3+1$ dimensions carries three configuration directions per point and hence a six-dimensional phase space. By contrast, the intrinsic gravitational data on a null hypersurface are specified on the two-dimensional surface Σ, whose metric h_{AB} has only two independent components per point, together with the corner scalar sector. The null direction itself is tangent to $\mathcal{N}$ and does not furnish an additional independent canonical direction. Consequently, the kinematic null-boundary phase space contains two canonical pairs per Planck-area element, corresponding to four phase-space directions,

$$d_{\text{naive}} = 4. \tag{4}$$

IV. TWO GEOMETRIC REDUCTIONS

We count degrees of freedom *per Planck-area element* on a two-dimensional cross-section $\Sigma \subset \mathcal{N}$. The intrinsic gravitational data on a null hypersurface therefore live on Σ, not on a three-dimensional spatial slice. We define the *symplectic rank density* d as the number of independent phase-space directions per Planck-area cell. The number of canonical pairs per cell is then

$$n_{\text{pair}} = \frac{d}{2}. \tag{5}$$

From the null-boundary symplectic form (3), the kinematic null data contain two canonical pairs per cell (shear sector and scalar area/boost sector), hence

$$n_{\text{pair}}^{\text{naive}} = 2, \qquad d_{\text{naive}} = 4. \tag{6}$$

A. Null constraint surface

On a quasi-stationary horizon segment, the radiative/shear sector may be treated as fixed flux data. The null characteristic constraint then removes one canonical pair density relative to the kinematic null data. Since one canonical pair corresponds to two phase-space directions, the symplectic rank density is reduced by two:

$$d_1 = d_{\text{naive}} - 2 = 2, \qquad n_{\text{pair}}^{(1)} = 1. \tag{7}$$

B. Exterior symplectic quotient

Exterior observables are insensitive to variations supported purely in the interior. Such directions lie in the kernel of the restricted symplectic form and must be quotiented out:

$$\mathcal{P}_{\text{ext}} = \frac{\mathcal{C}}{\ker \Omega_{\text{ext}}}. \tag{8}$$

In the remaining scalar sector (A, η), variations of the boost variable η generated by changes of interior anchoring have vanishing pairing with all exterior-supported variations. The quotient therefore removes one additional phase-space direction density,

$$d_{\text{eff}} = d_1 - 1 = 1, \qquad n_{\text{pair}} = \frac{d_{\text{eff}}}{2} = \frac{1}{2}. \tag{9}$$

V. LIOUVILLE SCALING AND THE 1/4 COEFFICIENT

The Liouville measure is constructed from the top exterior power of the symplectic form, $d\mu_L = \omega^N/N!$, where N is the number of canonical pairs. For a phase-space density defined per Planck-area element on a two-dimensional cut, the exponent governing the scaling of accessible Liouville volume is therefore proportional to the canonical-pair density $n_{\rm pair}$, with an additional factor 1/2 arising from the relation between phase-space dimension and number of pairs. Hence

$$\ln \Omega_{\rm acc} \sim \frac{n_{\rm pair}}{2} \frac{A}{\ell_P^2}. \tag{10}$$

With $n_{\rm pair} = 1/2$,

$$\ln \Omega_{\rm acc} \sim \frac{1}{4} \frac{A}{\ell_P^2}, \qquad S = k_B \ln \Omega_{\rm acc} = \frac{k_B A}{4\ell_P^2}. \tag{11}$$

VI. DISCUSSION

The 1/4 factor emerges as a property of null-boundary symplectic geometry: a null constraint reduces the kinematic rank, and an exterior quotient reduces it further. The result is independent of microscopic model assumptions.

It is instructive to contrast this geometric origin with the familiar Hamiltonian thermodynamic picture. Globally, the gravitational–matter system evolves via Hamiltonian flow and fine-grained Liouville volume is conserved. Entropy growth in an exterior description prior to horizon formation can be understood as arising from restricted access to the global phase space, but such dynamical considerations depend on model details and do not fix a universal coefficient.

Once a null boundary is present, the restriction to an exterior observable algebra becomes tied to a universal geometric structure: the null-boundary symplectic form and its constraint-induced reductions. The resulting symplectic quotient is an algebraic/geometric operation rather than a consequence of dynamical mixing. In this sense, the horizon result may be viewed as a geometric fixed point of the more general idea that entropy reflects restricted phase-space accessibility.

A suggestive perspective emerges upon quantization. In the classical theory, the exterior restriction is implemented as a symplectic quotient by kernel directions. Upon quantization, this corresponds to the familiar purification structure: a mixed state on a subsystem arises from tracing over complementary degrees of freedom in a larger Hilbert space. The interior null data thus act as a minimal "purifier" for the exterior sector. We do not pursue this correspondence further here, but it suggests that the necessity and size of the purifier are already encoded in classical null-boundary phase space.

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[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[3] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[4] J. Lee and R. M. Wald, J. Math. Phys. **31**, 725 (1990).
[5] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[6] F. Hopfmüller and L. Freidel, Phys. Rev. D **97**, 124029 (2018).
[7] L. Ciambelli, R. G. Leigh, and P.-C. Pai, JHEP **01**, 020 (2023).
[8] A. Speranza, JHEP **02**, 021 (2018).
[9] W. Donnelly and L. Freidel, JHEP **09**, 102 (2016).
[10] V. Chandrasekaran and É. É. Flanagan, "Subregion algebras in classical and quantum gravity," arXiv:2601.07915 [gr-qc] (2026).
[11] L. D. Landau and E. M. Lifshitz, *Statistical Physics.*
[12] R. K. Pathria and P. D. Beale, *Statistical Mechanics.*